Pergamon

*Int. J. Solids Structures* Vol. 34, No. 10, pp. 1183–1197, 1997

0020–7683/97 $17.00 + .00

**PII: S0020–7683(96)00104–7**

# AN ASYMPTOTIC ANALYSIS TO A TENSILE CRACK IN CREEPING SOLIDS COUPLED WITH DAMAGE—PART II. LARGE DAMAGE REGION VERY NEAR THE CRACK TIP

M. LU, S. B. LEE†, J. Y. KIM
Department of Mechanical Engineering, Korea Advanced Institute of Science and Technology, 373-1 Kusong-dong, Yusong-gu, Taejon 305–701, Korea

and

H. C. MAI
Institute of Solid Mechanics, Beijing University of Aeronautics and Astronautics, Beijing 100083, China

(*Received* 24 *October* 1995; *in revised form* 4 *June* 1996)

**Abstract**—A coordinate perturbation approach is used to deal with the asymptotic behavior of damage in the region very near the tip of a static mode I crack in creeping solids. Like Part I, the damage effect here is also incorporated into the power-law viscous creep constitutive equations by using the strain equivalence principle and the evolution of the cumulative damage is described by the multi-axial Kachanov–Rabotnov kinetics equation. The equation derived poses a nonlinear eigenvalue problem which has to be solved by numerical approaches. The solution obtained enables one to realize how the damage has effect on the crack tip field and what manner stresses vary with when the crack tip is quite closely approached. Examples are given to illustrate the distributions of stresses, strains and damage. © 1997 Elsevier Science Ltd. All rights reserved.

## 1. INTRODUCTION

In Part I of this context (Lee *et al.*, 1996) the authors worked out a procedure to analyze the effect of damage in the field near a crack tip, where a small damage condition holds. The solution obtained in that study is featured in a HRR singularity with the damage as a perturbation, and it does not apply to the regime where the crack tip is quite approached and large damage occurs therein. As is known, the HRR singularity contains a certain anomaly which predicts infinite stresses at the crack tip. However, the singularity field does not persist all the way if more factors are taken into account. For example, within the consideration of large deformation, the crack tip will be blunted because of large strains, which reduces the stress substantially. In fact, one of the components of stress, $\sigma_{xx}$, must vanish rather than going to infinity as the blunted crack tip is a free surface (McMeeking and Parks, 1979). On the other hand, according to continuum damage mechanics concept, a rupture process of media is related to a certain damage behavior and the ultimate state of damage is the local, or even global, fracture of a body. For materials with a stationary crack in creeping and undergoing cumulatively damaging, there exist two competing effects around the crack tip, one is the stress concentration tendency arising from the singularity of stress and another is the stress relaxation caused by damage. In the absence of the damage effect, analyses lead to a stress singularity and this case has been investigated in the pioneering literature for power-law creep law materials (see, Hutchinson, 1968; Rice and Rosengren, 1968; Riedel and Rice, 1980). However, while damaging phenomena take place in a material, they will result in the relaxation of stresses. The greater the stresses are, or the longer the loading duration is, the larger the damage becomes and the larger damage will in turn reduce the stresses significantly. It is to be expected during such a coupling process that the damage has great effect on the stress singularity of a cracked body and

† Author to whom correspondence should be addressed.

even, as shown later in the context, when the damage involved reaches its critical value (say, a damage measure reaches unity) at the crack tip and a complete fracture failure occurs. In this case stresses are relaxed to vanishing. Following such an idea, Li *et al.* (1988) and Bassani and Hawk (1990), for instance, used numerical procedures to analyze the influence of the damage on the crack tip field while several other researchers (e.g. Gao, 1986, 1995), based on some theoretical models, give analytical estimations to damage field around the crack tip.

In this paper we use a coordinate perturbation approach to study the stress and strain distribution analytically in the limiting case in which the region concerned is very near the crack tip where damage is very severe. From the analytic structure of the asymptotic solution one can realize how the damage material constants dominate the crack behavior of the medium and, therefore, attain a pattern reflecting the transition from a partially damaged state to a complete failure, namely, fracture of the medium, in a time dependent field around a static crack in creeping materials.

As in part I, since analysis, including the instantaneously elastic effect, is beyond the scope of what we are currently able to do, our attention is limited to the case in which creep damage has the dominant effect.

We first list all the governing equations and the corresponding initial and boundary conditions in Section 2 and an asymptotic form is proposed in Section 3. In this section coordinate perturbation expansion is conducted based on the asymptotic form and the corresponding asymptotic equations are subsequently derived. An example, the first-order approximation, is given in detail. Results are shown in Section 4 and discussion is held on them. Remarks and notes are made in Section 5.

## 2. GOVERNING EQUATIONS AND INITIAL AND BOUNDARY CONDITIONS

For a mode I crack problem under the plane stress condition, if designating a polar coordinate system $(r, \theta)$ with $\theta = 0$ directly ahead of the crack and the origin at the crack tip, the fundamental equation can be proposed as follows.

### 2.1. *Stress function*

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right), \quad \sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}, \tag{1}$$

where $\phi$ denotes the Airy stress potential and $\sigma_{ij}$ $(i,j = r, \theta)$ are the stresses.

### 2.2. *Compatibility equation*

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\dot{\varepsilon}_{\theta\theta}) + \frac{1}{r^2}\frac{\partial^2 \dot{\varepsilon}_{rr}}{\partial \theta^2} - \frac{1}{r}\frac{\partial \dot{\varepsilon}_{rr}}{\partial r} - \frac{2}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial \dot{\varepsilon}_{r\theta}}{\partial \theta}\right) = 0, \tag{2}$$

where $\varepsilon_{ij}$ $(i,j = r, \theta)$ label strains and a dot over them indicates the derivative with respect to time, i.e.

$$\dot{\varepsilon}_{ij} = \frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_{ij}. \tag{3}$$

### 2.3. *Constitutive equations*

Neglecting the effect of elasticity, the power-law viscous creep materials coupled with damage can be described as (Lee *et al.*, 1996 for details)

$$\dot{\varepsilon}_{\theta\theta} = \frac{3}{2}\frac{B}{(1-\omega)^n} s_{\theta\theta}\bar{\sigma}^{n-1}, \tag{4}$$

$$\dot{\varepsilon}_{r\theta} = \frac{3}{2}\frac{B}{(1-\omega)^n} s_{r\theta}\bar{\sigma}^{n-1}, \tag{5}$$

$$\dot{\varepsilon}_{rr} = \frac{3}{2}\frac{B}{(1-\omega)^n} s_{rr}\bar{\sigma}^{n-1}, \tag{6}$$

where $B$ and $n$ are material coefficients dependent on temperature, $\omega$ is a damage variable, $s_{ij}$ $(i,j = r, \theta)$ represent the deviatoric stresses and $\bar{\sigma}$ indicates the equivalent stress defined as

$$\bar{\sigma} = [\sigma_{rr}^2 + \sigma_{\theta\theta}^2 - \sigma_{rr}\sigma_{\theta\theta} + 3\sigma_{r\theta}^2]^{1/2}. \tag{7}$$

2.4. *Damage evolution law*

For a creeping material the cumulative damage evolution law can be described by the Kachanov–Rabotnov equation (e.g. Kachanov, 1986; Rabotnov, 1969; Hayhurst, 1972 and Chaboche, 1988a, b)

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \left[\frac{\chi(\sigma)}{A}\right]^{\mu}(1-\omega)^{-k} \tag{8}$$

or its integral form

$$\omega = 1 - \left\{1-(k+1)\int_0^t \left[\frac{\chi(\sigma)}{A}\right]^{\mu} \mathrm{d}\tau\right\}^{1/k+1}, \tag{9}$$

in which

$$\chi(\sigma) = \alpha J_0(\sigma) + \beta J_1(\sigma) + (1-\alpha-\beta)\sqrt{J_2(\sigma)}, \tag{10}$$

where, $J_0 = \sigma_1$ represents the principal stress, $J_1 = \mathrm{tr}\sigma$ indicates the hydrostatic stress and $J_2 = 1/2\ s_{ij}s_{ij}$ is the second invariant of deviatoric stresses, while $A$, $\mu$, $k$, $\alpha$ and $\beta$ are material constants. Here, $A > 0$ and $\mu \geqslant 1$ (see Kachanov, 1986). Particularly, when $\mu = k$, eqn (8) will lead to the earlier form of the Kachanov damage evolution law. Besides, experimental data show that for many ductile materials such as aluminum etc., $\alpha \approx 0$ and $\beta \approx 0$ (Hayhurst, 1972).

As is shown, the damage model (8) leads to an accumulative effect with time going by and finally will result in complete failure after undergoing a critical time duration, say, the rupture time $t_F$ (note that $t_F$ is a loading, also temperature and geometry dependent parameter in practice), namely,

$$\int_0^1 (1-\omega)^k\,\mathrm{d}\omega = \int_0^{t_F}\left[\frac{\chi[\sigma]}{A}\right]^{\mu}\mathrm{d}\tau, \tag{11}$$

which leads to

$$(k+1)\int_0^{t_F}\left[\frac{\chi[\sigma]}{A}\right]^{\mu} = 1 \tag{12}$$

provided that $\chi[\sigma]$ does not vanish identically. Thus, the damage evolution law (9) can be alternatively written as

$$\omega = 1 - \left\{ (k+1) \int_t^{t_F} \left[ \frac{\chi(\sigma)}{A} \right]^{\mu} d\tau \right\}^{1/k+1}. \tag{13}$$

The initial condition is that a load is applied suddenly to the cracked specimen at the time $t = 0$. Since we are concerned with the creep damage, the elastic effect which leads to an instantaneous response of the material will be ignored.

Boundary conditions are prescribed on the traction-free crack faces, $n_i\sigma_{ij} = 0$ ($n_i$ = normal vector on the crack face) and at infinity.

## 3. LOCAL FORM OF AN ASYMPTOTIC SOLUTION

Since our consideration is to the local asymptotic solution in the zone very near the crack tip, it is proposed, following a coordinate perturbation procedure that the solution takes such a form

$$\phi(r, \theta, t) = r_s \sum_{m=0}^{\infty} r^m \phi_m(\theta, t), \tag{14}$$

where $s$ is an undetermined constant. Substitution of (14) into (1) one is left with

$$\sigma_{\theta\theta}(r, \theta, t) = r^{s-2} \sum_{m=0}^{\infty} (m+s)(s+m-1)\phi_m(\theta, t)r^m, \tag{15}$$

$$\sigma_{r\theta}(r, \theta, t) = -r^{s-2} \sum_{m=0}^{\infty} (s+m-1)\phi'_m(\theta, t)r^m, \tag{16}$$

$$\sigma_{rr}(r, \theta, t) = r^{s-2} \sum_{m=0}^{\infty} [(m+s)\phi_m(\theta, t) + \phi''_m(\theta, t)]r^m. \tag{17}$$

Since at the crack tip the medium is supposed to be completely damaged and thereby cannot sustain stresses in accordance with the concept of damage mechanics, we then have

$$\lim_{r \to 0} \sigma_{ij} = 0 \quad (i, j = 1, 2), \tag{18}$$

which obviously require that

$$s > 2. \tag{19}$$

In the following discussion for convenience let

$$I_1(r, \theta, t) = \frac{1}{3} \sum_{m=0}^{\infty} [(m+s)^2 \phi_m(\theta, t) + \phi''_m(\theta, t)]r^m \tag{20}$$

and

$$I_2(r, \theta, t) = \left\{ \left[ \sum_{m=0}^{\infty} [(m+s)\phi_m(\theta, t) + \phi''_m(\theta, t)]r^m \right]^2 + \left[ \sum_{m=0}^{\infty} [(m+s)(m+s-1)\phi_m(\theta, t)]r^m \right]^2 \right.$$
$$- \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} [(p+s)(p+s-1)[(q+s)\phi_q(\theta, t) + \phi''_q(\theta, t)]\phi_p(\theta, t)]r^{p+q}$$
$$\left. + 3\left[ \sum_{m=0}^{\infty} (s+m-1)\phi'_m(\theta, t)r^m \right]^2 \right\}^{1/2}. \tag{21}$$

Thus, we can denote that

$$J_1(r,\theta,t) = r^{s-2}\mathrm{I}_1(r,\theta,t) \tag{22}$$

$$\bar{\sigma}(r,\theta,t) = r^{s-2}\mathrm{I}_2(r,\theta,t). \tag{23}$$

It can be easily shown through inserting eqns (15)–(17) into eqn (10) and using the expressions of (22) and (23) that the damage evolution law can be rewritten as

$$\omega(r,\theta,t) = 1-(k+1)^{1/k+1}A^{-\mu/k+1}r^{\mu(s-2)/k+1}\Omega(r,\theta,t) \tag{24}$$

where

$$\Omega(r,\theta,t) = \left\{\int_t^{t_F}\left[\beta\mathrm{I}_1(r,\theta,\tau)+\frac{1}{\sqrt{3}}(1-\beta)\mathrm{I}_2(r,\theta,\tau)\right]^{\mu}\mathrm{d}\tau\right\}^{1/k+1}. \tag{25}$$

From the relation

$$s_{ij} = \sigma_{ij}-\frac{1}{3}\sigma_{kk}\delta_{ij}, \quad (i,j,k = r,\theta) \tag{26}$$

and substituting eqns (15)–(17) into eqn (26) we have

$$s_{\theta\theta}(r,\theta,t) = \frac{1}{3}r^{s-2}\Lambda_{\theta\theta}(r,\theta,t), \tag{27}$$

$$s_{r\theta}(r,\theta,t) = \frac{1}{3}r^{s-2}\Lambda_{r\theta}(r,\theta,t), \tag{28}$$

$$s_{rr}(r,\theta,t) = \frac{1}{3}r^{s-2}\Lambda_{rr}(r,\theta,t), \tag{29}$$

where

$$\Lambda_{\theta\theta} = \sum_{m=0}^{\infty}[(m+s)(2s+2m-3)\phi_m(\theta,t)-\phi_m''(\theta,t)]r^m, \tag{30}$$

$$\Lambda_{r\theta} = -3\sum_{m=0}^{\infty}(s+m-1)\phi_m'(\theta,t)r^m, \tag{31}$$

$$\Lambda_{rr} = \sum_{m=0}^{\infty}[(m+s)(3-m-s)\phi_m(\theta,t)+2\phi_m''(\theta,t)]r^m. \tag{32}$$

Thus, for eqns (4)–(6) one has

$$\dot{\varepsilon}_{\theta\theta}(r,\theta,t) = \tfrac{1}{2}B(k+1)^{-n/k+1}A^{n\mu/k+1}r^{n(s-2)(1-\mu/k+1)}\Lambda_{\theta\theta}(r,\theta,t)\mathrm{I}_2^{n-1}(r,\theta,t)[\Omega^n(r,\theta,t)]^{-1}, \tag{33}$$

$$\dot{\varepsilon}_{r\theta}(r,\theta,t) = \tfrac{1}{2}B(k+1)^{-n/k+1}A^{n\mu/k+1}r^{n(s-2)(1-\mu/k+1)}\Lambda_{r\theta}(r,\theta,t)\mathrm{I}_2^{n-1}(r,\theta,t)[\Omega^n(r,\theta,t)]^{-1}, \tag{34}$$

$$\dot{\varepsilon}_{rr}(r,\theta,t) = \tfrac{1}{2}B(k+1)^{-n/k+1}A^{n\mu/k+1}r^{n(s-2)(1-\mu/k+1)}\Lambda_{rr}(r,\theta,t)\mathrm{I}_2^{n-1}(r,\theta,t)[\Omega^n(r,\theta,t)]^{-1}, \tag{35}$$

letting

$$\Gamma_{\theta\theta}(r,\theta,t) = \Lambda_{\theta\theta}(r,\theta,t)\mathrm{I}_2^{n-1}(r,\theta,t)[\Omega^n(r,\theta,t)]^{-1}, \tag{36}$$

$$\Gamma_{r\theta}(r,\theta,t) = \Lambda_{r\theta}(r,\theta,t)\,\mathrm{I}_2^{n-1}(r,\theta,t)[\Omega^n(r,\theta,t)]^{-1}, \tag{37}$$

$$\Gamma_{rr}(r,\theta,t) = \Lambda_{rr}(r,\theta,t)\mathrm{I}_2^{n-1}(r,\theta,t)[\Omega^n(r,\theta,t)]^{-1}, \tag{38}$$

and expanding eqns (36)–(38) into the Taylor series with respect to $r$, one has

$$\Gamma_{\theta\theta}(r,\theta,t) = \Gamma_{\theta\theta}^{(0)}(\theta,t)+\Gamma_{\theta\theta}^{(1)}(\theta,t)r+\tfrac{1}{2!}\Gamma_{\theta\theta}^{(2)}(\theta,t)r^2+,\dots, \tag{39}$$

$$\Gamma_{r\theta}(r,\theta,t) = \Gamma_{r\theta}^{(0)}(\theta,t)+\Gamma_{r\theta}^{(1)}(\theta,t)r+\tfrac{1}{2!}\Gamma_{r\theta}^{(2)}(\theta,t)r^2+,\dots, \tag{40}$$

$$\Gamma_{rr}(r,\theta,t) = \Gamma_{rr}^{(0)}(\theta,t)+\Gamma_{rr}^{(1)}(\theta,t)r+\tfrac{1}{2!}\Gamma_{rr}^{(2)}(\theta,t)r^2+,\dots \tag{41}$$

Correspondingly,

$$\dot{\varepsilon}_{\theta\theta}(r,\theta,t) = \tfrac{1}{2}B(k+1)^{-n/k+1}A^{n\mu/k+1}r^{n(s-2)(1-\mu/k+1)} \cdot[\Gamma_{\theta\theta}^{(0)}(\theta,t)+\Gamma_{\theta\theta}^{(1)}(\theta,t)r+\tfrac{1}{2!}\Gamma_{\theta\theta}^{(2)}(\theta,t)r^2+,\dots,], \tag{42}$$

$$\dot{\varepsilon}_{r\theta}(r,\theta,t) = \tfrac{1}{2}B(k+1)^{-n/k+1}A^{n\mu/k+1}r^{n(s-2)(1-\mu/k+1)} \cdot[\Gamma_{r\theta}^{(0)}(\theta,t)+\Gamma_{r\theta}^{(1)}(\theta,t)r+\tfrac{1}{2!}\Gamma_{r\theta}^{(2)}(\theta,t)r^2+,\dots,], \tag{43}$$

$$\dot{\varepsilon}_{rr}(r,\theta,t) = \tfrac{1}{2}B(k+1)^{-n/k+1}A^{n\mu/k+1}r^{n(s-2)(1-\mu/k+1)} \cdot[\Gamma_{rr}^{(0)}(\theta,t)+\Gamma_{rr}^{(1)}(\theta,t)r+\tfrac{1}{2!}\Gamma_{rr}^{(2)}(\theta,t)r^2+,\dots,]. \tag{44}$$

Substitution of eqns (42)–(44) into eqn (2) yields

$$\begin{aligned}
&\frac{1}{r}\frac{\partial^2}{\partial r^2}\left\{r^{v+1}\left[\Gamma_{\theta\theta}^{(0)}(\theta,t)+\Gamma_{\theta\theta}^{(1)}(\theta,t)r+\frac{1}{2!}\Gamma_{\theta\theta}^{(2)}(\theta,t)r^2+,\dots,\right]\right\}\\
&+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left\{r^{v}\left[\Gamma_{rr}^{(0)}(\theta,t)+\Gamma_{rr}^{(1)}(\theta,t)r+\frac{1}{2!}\Gamma_{rr}^{(2)}(\theta,t)r^2+,\dots,\right]\right\}\\
&-\frac{1}{r}\frac{\partial}{\partial r}\left\{r^{v}\left[\Gamma_{rr}^{(0)}(\theta,t)+\Gamma_{rr}^{(1)}(\theta,t)r+\frac{1}{2!}\Gamma_{rr}^{(2)}(\theta,t)r^2+,\dots,\right]\right\}\\
&-\frac{2}{r^2}\frac{\partial}{\partial r}\left\{r^{v+1}\frac{\partial}{\partial\theta}\left[\Gamma_{r\theta}^{(0)}(\theta,t)+\Gamma_{r\theta}^{(1)}(\theta,t)r+\frac{1}{2!}\Gamma_{r\theta}^{(2)}(\theta,t)r^2+,\dots,\right]\right\}=0,
\end{aligned} \tag{45}$$

in which

$$v = n(s-2)\left(1-\frac{\mu}{k+1}\right). \tag{46}$$

Equalizing the terms with the same order power, we have

$$\left[\nu(\nu+1)\Gamma_{\theta\theta}^{(0)}(\theta,t)-\nu\Gamma_{rr}^{(0)}(\theta,t)-2(\nu+1)\frac{\partial}{\partial\theta}\Gamma_{r\theta}^{(0)}(\theta,t)+\frac{\partial^2}{\partial\theta^2}\Gamma_{rr}^{(0)}(\theta,t)\right]r^{\nu-2}$$

$$+\left[(\nu+2)(\nu+1)\Gamma_{\theta\theta}^{(1)}(\theta,t)-(\nu+1)\Gamma_{rr}^{(1)}(\theta,t)-2(\nu+2)\frac{\partial}{\partial\theta}\Gamma_{r\theta}^{(1)}(\theta,t)+\frac{\partial^2}{\partial\theta^2}\Gamma_{rr}^{(1)}(\theta,t)\right]r^{\nu-1}$$

$$+\frac{1}{2}\left[(\nu+2)(\nu+3)\Gamma_{\theta\theta}^{(2)}(\theta,t)-(\nu+2)\Gamma_{rr}^{(2)}(\theta,t)-2(\nu+3)\frac{\partial}{\partial\theta}\Gamma_{r\theta}^{(2)}(\theta,t)+\frac{\partial^2}{\partial\theta^2}\Gamma_{rr}^{(2)}(\theta,t)\right]r$$

$$+,\dots,=0, \tag{47}$$

such that

$$\nu(\nu+1)\Gamma_{\theta\theta}^{(0)}(\theta,t)-\nu\Gamma_{rr}^{(0)}(\theta,t)-2(\nu+1)\frac{\partial}{\partial\theta}\Gamma_{r\theta}^{(0)}(\theta,t)+\frac{\partial^2}{\partial\theta^2}\Gamma_{rr}^{(0)}(\theta,t)=0, \tag{48}$$

$$(\nu+2)(\nu+1)\Gamma_{\theta\theta}^{(1)}(\theta,t)-(\nu+1)\Gamma_{rr}^{(1)}(\theta,t)-2(\nu+2)\frac{\partial}{\partial\theta}\Gamma_{r\theta}^{(1)}(\theta,t)+\frac{\partial^2}{\partial\theta^2}\Gamma_{rr}^{(1)}(\theta,t)=0, \tag{49}$$

$$\frac{1}{2}(\nu+2)(\nu+3)\Gamma_{\theta\theta}^{(2)}(\theta,t)-\frac{1}{2}(\nu+2)\Gamma_{rr}^{(2)}(\theta,t)-(\nu+3)\frac{\partial}{\partial\theta}\Gamma_{r\theta}^{(2)}(\theta,t)+\frac{1}{2}\frac{\partial^2}{\partial\tau^2}\Gamma_{rr}^{(2)}(\theta,t)=0. \tag{50}$$

In this way, we obtain a series of asymptotic equations. Letting

$$\phi_m(\theta,t)=\rho_m(t)\Theta_m(\theta),\quad m=0,1,2,\dots, \tag{51}$$

the asymptotic equations (48)–(50) can be reduced to a family of ordinary differential equations with the time-dependent functions $\{\rho_m(t)\}$ remaining undetermined.

Based on the decomposition (51) for $\phi_m(\theta,t)$, for the first-order approximation (48) one has

$$R^{n-1}[\Theta_0(\theta)]Q^{-n\mu/k+1}[\Theta_0]\{\nu(\nu+1)P_2[\Theta_0(\theta)]-\nu P_1[\Theta_0(\theta)]\}$$
$$-2(\nu+1)\frac{\mathrm{d}}{\mathrm{d}\theta}\{P_3[\Theta_0(\theta)]R^{n-1}[\Theta_0(\theta)]Q^{-n\mu/k+1}[\Theta_0(\theta)]\}$$
$$+\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\{P_1[\Theta_0(\theta)]R^{n-1}[\Theta_0(\theta)]Q^{-n\mu/k+1}[\Theta_0(\theta)]\}=0, \tag{52}$$

in which

$$P_1=[s(3-s)\Theta_0+2\Theta_0''], \tag{53}$$

$$P_2=[s(2s-3)\Theta_0-\Theta_0''], \tag{54a}$$

$$P_3=[(3-3s)\Theta_0'], \tag{54b}$$

$$R=\{[s\Theta_0+\Theta_0'']^2+[s(s-1)\Theta_0]^2-s(s-1)\Theta_0[s\Theta_0+\Theta_0'']+3[(s-1)\Theta_0']^2\}^{1/2}, \tag{55}$$

$$Q=\left\{\beta\left[s^2\Theta_0+\Theta_0''-\frac{1}{\sqrt{3}}R\right]+\frac{1}{\sqrt{3}}R\right\} \tag{56}$$

and the stresses are

$$\sigma_{\theta\theta}^{(1)}(r,\theta,t) = \rho_0(t)r^{s-2}[s(s-1)\Theta_0(\theta)]. \tag{57}$$

$$\sigma_{r\theta}^{(1)}(r,\theta,t) = \rho_0(t)r^{s-2}[(s-1)\Theta_0'(\theta)], \tag{58}$$

$$\sigma_{rr}^{(1)}(r,\theta,t) = \rho_0(t)r^{s-2}[s\Theta_0(\theta)+\Theta_0''(\theta)]. \tag{59}$$

From eqn (13) one can obtain the damage distribution

$$\omega = 1-A^{-\mu/k+1}(k+1)^{1/k+1}\left[\int_t^{t_F}[\rho_0(\tau)]^{\mu}\,d\tau\right]^{1/k+1} r^{\mu(s-2)/k+1}\Omega^{(0)}(\theta), \tag{60}$$

where

$$\Omega^{(0)}(\theta) = \left[\beta I_1^{(0)}(\theta)+\frac{1}{\sqrt{3}}(1-\beta)I_2^{(0)}(\theta)\right]^{\mu/k+1} \tag{61}$$

in which

$$I_1^{(0)}(\theta) = s^2\Theta_0(\theta)+\Theta_0''(\theta) \tag{62}$$

and

$$I_2^{(0)}(\theta) = \{[s\Theta_0(\theta)+\Theta_0''(\theta)]^2+[s(s-1)\Theta_0(\theta)]^2 \\ -s(s-1)[s\Theta_0(\theta)+\Theta_0''(\theta)]\Theta_0(\theta)+(s-1)\Theta_0'(\theta)]^2\}^{1/2}. \tag{63}$$

Correspondingly, in terms of eqns (33)–(35) the strains can be found to be

$$\dot{\varepsilon}_{\theta\theta}(r,\theta,t) = \frac{1}{2}B(k+1)^{-n/k+1}A^{n\mu/k+1}r^{n(s-2)(1-\mu/k+1)}\Lambda_{\theta\theta}^{(0)}(\theta)[I_2^{(0)}(\theta)]^{n-1}[\Omega^{(0)}(\theta)]^{-n}, \tag{64}$$

$$\dot{\varepsilon}_{r\theta}(r,\theta,t) = \frac{1}{2}B(k+1)^{-n/k+1}A^{n\mu/k+1}r^{n(s-2)(1-\mu/k+1)}\Lambda_{r\theta}^{(0)}(\theta)[I_2^{(0)}(\theta)]^{n-1}[\Omega^{(0)}(\theta)]^{-n}, \tag{65}$$

$$\dot{\varepsilon}(r,\theta,t) = \frac{1}{2}B(k+1)^{-n/k+1}A^{n\mu/k+1}r^{n(s-2)(1-\mu/k+1)}\Lambda_{rr}^{(0)}(\theta)[I_2^{(0)}(\theta)]^{n-1}[\Omega^{(0)}(\theta)]^{-n}, \tag{66}$$

where

$$\Lambda_{\theta\theta}^{(0)}(\theta) = s(2s-3)\Theta_0(\theta)-\Theta_0''(\theta), \tag{67}$$

$$\Lambda_{r\theta}^{(0)}(\theta) = -3(s-1)\Theta_0'(\theta), \tag{68}$$

$$\Lambda_{rr}^{(0)}(\theta) = s(3-s)\Theta_0+2\Theta_0''(\theta). \tag{69}$$

Starting from eqn (49) one can obtain the second-order approximation and following the same procedure higher-order approximations can also be arrived at. The corresponding boundary conditions are as follows

$$\Theta_m'(0) = \Theta_m'''(0) = 0, \quad \Theta_m(\pi) = \Theta_m'(\pi) = 0, \quad m = 0,1,2,3,\ldots,. \tag{70}$$

The time-dependent functions $\{\rho_m(t)\}$, which depend upon the applied loading and the geometry of cracked body, cannot be obtained merely by virtue of the present asymptotic analysis and a global solution is needed to determine it rigorously. Obviously, since it seems impossible to get a global analytical solution for the problem described by eqns (1)–(6), an approximate estimation of them or some numerical results are expected to be useful.

## 4. RESULTS AND DISCUSSIONS

Note that the constant $s$ has remained undetermined so far. In fact, $s$ is an eigenvalue which depends upon the material constants $n$, $k$ and $\mu$ and is designed by the nonlinear equation (52) and the boundary conditions (70) (when $m = 0$). It can be easily thought of that the energy condition

$$\lim_{r\to 0} \sigma_{ij}\varepsilon_{ij} \to O\left(\frac{1}{r}\right), \tag{71}$$

which has been extensively utilized by many researchers, could be invoked in this case. Combining (71), (15)–(17) and (33)–(35), one obtains

$$s = 2-\left[1+n\left(1-\frac{\mu}{k+1}\right)\right]^{-1}. \tag{72}$$

The relationship (72) provides a possible explicit form to determine one of the eigenvalues for our problem. However, whether the eigenvalue resolved by eqn (72) is a physical solution in the present case needs to further examine it. Towards this end, it can be seen from eqn (72) that there are three categories of $s$ for which the stress field may differ substantially, namely,

$$\left\{1+n\left[1-\frac{\mu}{k+1}\right]\right\}^{-1}\begin{cases}>0,\\=0,\\<0.\end{cases} \tag{73}$$

For the first case of eqn (73), it leads to

$$\mu < \frac{1}{n}(k+1)(n+1), \tag{74}$$

and correspondingly $s < 2$. From eqns (15)–(17) we can see that stresses then have a singularity with the order $O(1/r^{\lambda})$, where $\lambda = [1+n(1-\mu/k+1)]^{-1}$. Obviously, in this case both stresses and strains will go to infinity when $r$ approaches zero. This result is, however, contradictory to eqns (18) and (19) which reflect such a physical fact that a fully failed material mesovolume cannot sustain any loading and the stresses should vanish. Hence, this result is not what we require.

In the second case of eqn (73), the result will lead to some non-zero finite stress and strain components at the crack tip (note that the condition can be realized only when $n \to \infty$ or $\mu \to \infty$, whilst the former corresponds to the perfect-plasticity condition and the latter makes no practical sense). This situation does exist for a blunted crack tip since a free surface is therein created. However, since in this context no blunting concept is introduced and the crack tip remains a mathematical point, it is obviously unacceptable with a multi-value distribution of stresses and strains at a point. Thus, to make our solution be meaningful both mathematically and physically, this case can also be ruled out of our consideration.

Apparently, only the third case of eqn (73) could yield a result that satisfies the basic requirement designated by eqns (18) and (19). However, from great deal of numerical trials we found that the eigen equation (52), together with its corresponding boundary conditions, either only gives a trivial solution or results in greatly unstable calculations so that no solution can be obtained in this case.

We thus concluded that the tip energy condition (71) does not apply to our current analysis, as the eigenvalue $s$ determined by eqn (72), which can only be available within the range restricted by eqn (74), is not the required one which should satisfy the physical restraints (18) and (19). Hence, in order to seek the non-singular solution we work out here a numerical procedure to arrive at the eigenvalue $s$ and the corresponding eigensolution simultaneously. To begin with, we fix the stress potential $\Theta_0$ at $\theta = 0$ as unity, namely,

$$\Theta_0(0) = 1. \tag{75}$$

This condition can be realized by formally implementing some normalization procedure. Thus, eqn (52) and the boundary conditions (70) constitute a new two-point boundary-value problem with $s$ and $\phi''(0)$ being unknown.

In seeking the eigenvalue $s$ and the initial value $\Theta_0''(0)$ that satisfy the boundary conditions $\Theta_0(\pi) = \Theta_0''(\pi) = 0$, it is found that the solution property is quite complicated and very sensitive to initial iteration values. Actually, in order to obtain the eigenvalue $s$ and $\Theta_0''(0)$ we separately plot the root-curve of $\Theta_0(\pi) = F_1[s, \Theta_0''(0)] = 0$ and $\Theta_0'(\pi) = F_2[s, \Theta_0''(0)] = 0$ for a given set of $n$, $k$ and $\mu$. The intersection of the two curves give us the eigenvalue $s$ and $\Theta_0''(0)$ required. With this manner the two-point shooting problem can be decomposed into a one-point one. Once a solution is obtained for the given set of material constants, others can be proceeded by imposing a perturbation, for instance, with one or two thousandths of variation, to material constants and starting from the previous solutions as the initial iteration values.

Figures 1–3 illustrate some typical results of stress and strain distributions. In these figures the angular variations of stresses and strains are shown with $k = 2.2$, $\alpha = 0$, $\beta = 0$ and $\mu = 1.52$, but $n = 1, 2, 3$, respectively. Correspondingly, the eigenvalues that are sorted out with the presented approach are $s_1 = 4.7676$, $s_2 = 3.4466$ and $s_3 = 3.0766$, respectively. Note that in order to compare the variations of stress and strain in the same figures, we introduce scale factors in plotting. Evidently, this does not distort the required information since in angular distributions the physical quantities, whether stresses or strains, make sense merely with respect to their relative magnitudes. From Figs 1–3 we can see that the relative magnitude of $\sigma_{rr}$ and $\sigma_{\theta\theta}$ (and correspondingly $\varepsilon_{rr}$ and $\varepsilon_{\theta\theta}$) will substantially change with the increasing of $n$. That is, $\sigma_{rr}$ is smaller than $\sigma_{\theta\theta}$ at first and then closes to $\sigma_{\theta\theta}$ with the increasing of $n$ and finally becomes larger than $\sigma_{\theta\theta}$ while $n$ is larger than a certain value (this value depends on material constants $k$ and $\mu$). Another noteworthy phenomenon is

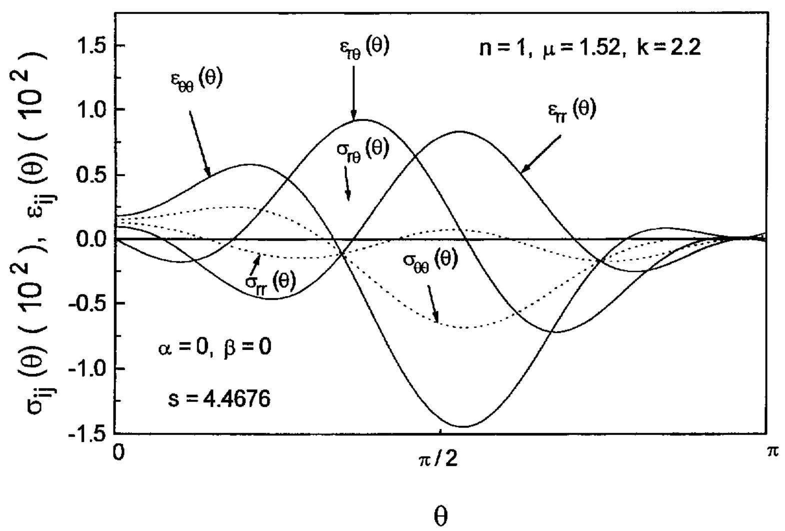


Fig. 1. The angular distributions of stresses and strains when $n = 1$, $\mu = 1.52$, $k = 2.2$, $\alpha = 0$ and $\beta = 0$. In this case the eigenvalue $s = 4.4676$.

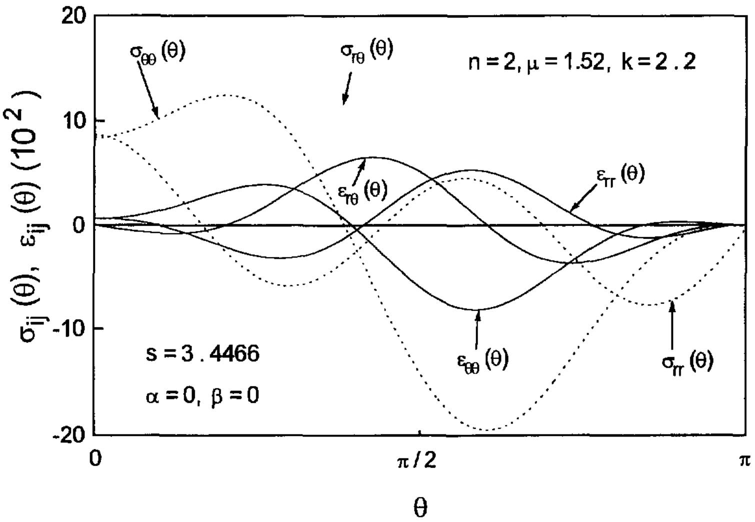


Fig. 2. The angular distributions of stresses and strains when $n = 2$, $\mu = 1.52$, $k = 2.2$, $\alpha = 0$ and $\beta = 0$. In this case the eigenvalue $s = 3.4466$.

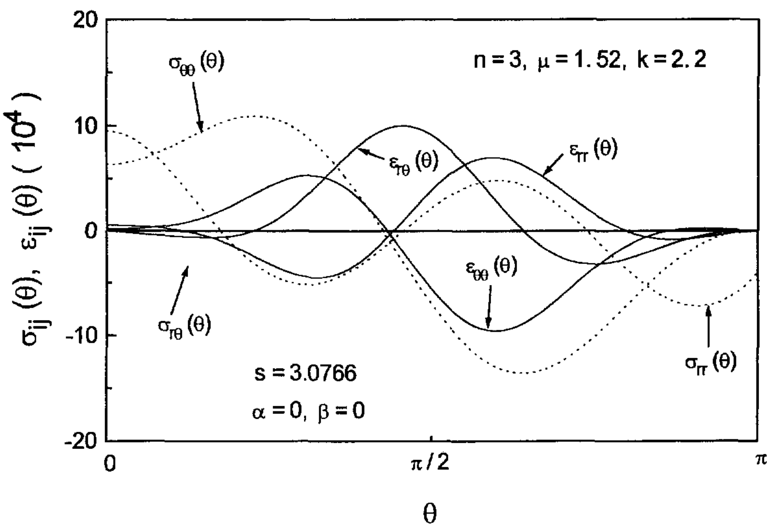


Fig. 3. The angular distributions of stresses and strains when $n = 3$, $\mu = 1.52$, $k = 2.2$, $\alpha = 0$ and $\beta = 0$. In this case the eigenvalue $s = 3.07655$.

that the angular distribution of stresses and strains exhibits approximately a quasi-periodic pattern and is generally different from that of the HRR problem.

Figures 4–6 illustrate the angular variation fields of stress when $n = 3$, $k = 2.2$, $\alpha = 0$, $\beta = 0$, but $\mu = 0$, but $\mu = 1.28$, 1.58 and 1.78, respectively. Here, the eigenvalues are $s_1 = 2.9630$, $s_2 = 3.1061$ and $s_3 = 3.2171$, respectively. It should be pointed out that in the first approximation of the asymptotic analysis $k$ and $\mu$ always appear in the combination of $\mu/k+1$, therefore the increasing of $\mu$ is equivalent to the correspondingly decreasing of $k$ and vice versa. A further analysis to the results demonstrated in Figs 4–6 may make one find that the variation of the angular stresses distribution with the increasing of the damage parameter $\mu$ is similar to that with the increasing of $n$. In fact, both $n$ and $\mu$ play the role to soften the material.

Figure 7 indicates the angular variation of $\Omega^0(\theta)$ with $n = 1, 2$ and 3, respectively, while $k = 2.2$, $\mu = 1.52$, $\alpha = 0$ and $\beta = 0$ and Fig. 8 shows the angular variation of $\Omega^0(\theta)$ with $n = 3$, $k = 2.2$, $\alpha = 0$ and $\beta = 1.28$, 1.58 and 1.78, respectively. Clearly, from (60) one is shown that $\Omega^0(\theta)$ actually denotes the angular variation portion of $(1-\omega)$, which represents a measure of an undamaged media in some way. Since in all the computations we take the

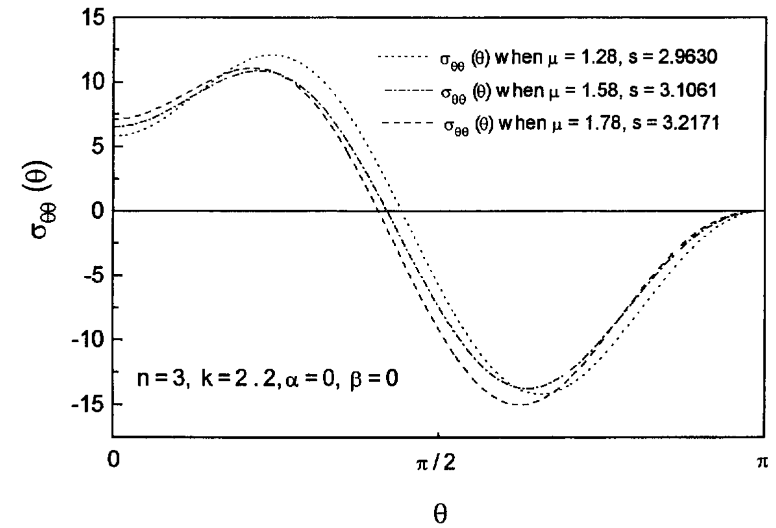


Fig. 4. The angular distributions of stress $\sigma_{\theta\theta}(\theta)$ when $n = 1, k = 2.2, \alpha = 0$ and $\beta = 0$, while $\mu = 1.28$, 1.58 and 1.78, with correspondingly the eigenvalue $s = 2.9630$, 3.1061 and 3.2171, respectively.

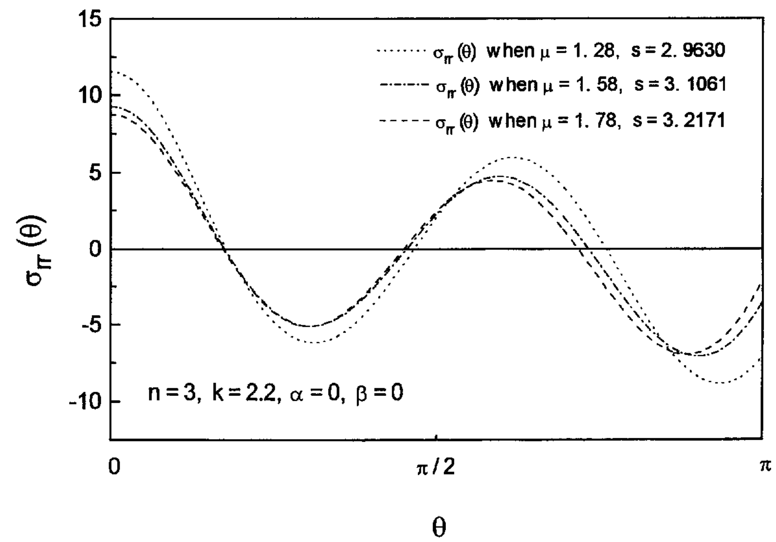


Fig. 5. The angular distributions of stress $\sigma_{rr}(\theta)$ when $n = 1, k = 2.2, \alpha = 0$ and $\beta = 0$, while $\mu = 1.28$, 1.58 and 1.78, with correspondingly the eigenvalue $s = 2.9630$, 3.1061 and 3.2171, respectively.

parameters $\alpha = 0$ to $\beta = 0$, this implies that the damage is essentially related to the equivalent stress $\bar{\sigma}(\theta)$. Through the results shown in Figs 7 and 8 one can easily see that the most severe damage primarily occurs at the place where $\theta = \pi$ or $\theta \approx \pi$ (say, in the case shown in Fig. 7 for $n = 1$, $k = 2.2$, $\mu = 1.52$, $\alpha = 0$ and $\beta = 0$). As the evolution of damage will finally lead to the full failure (rupture) of mesoelements of a material, the damage field has a substantial effect on the stress field of the tip. Besides, even though our analysis is performed to a stationary crack problem, it is to be expected that the distribution of damage obtained is able to supply additional hints in explaining some sophisticated experimental observations such as the variations of the direction of a crack extension, etc.

Note that all the results shown above exhibit that there always exist $s > 2$ and $v > 0$, the letter of which indicates that

$$\lim_{r \to 0} \dot{\varepsilon}_{ij} = 0. \tag{76}$$

Since the eigenvalue $s$ actually designates the rate of stresses tending to vanishing when the crack tip is approached, one could, although implicitly, find an interesting comparison with

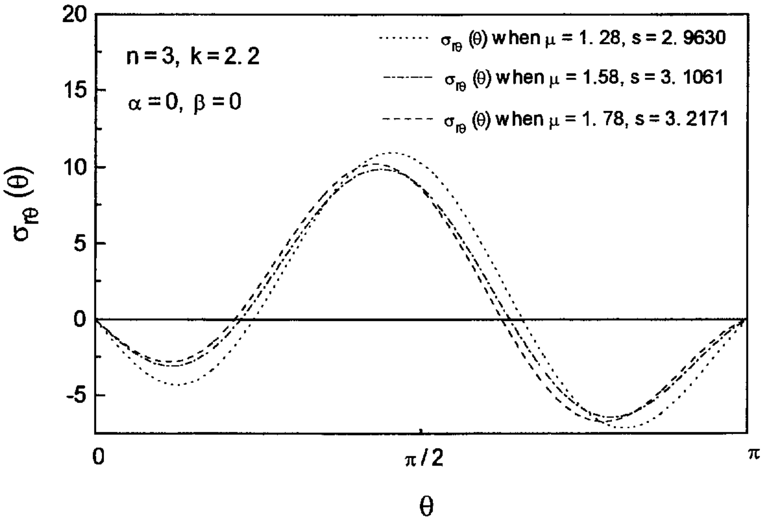


Fig. 6. The angular distributions of stress $\sigma_{r\theta}(\theta)$ when $n = 1$, $k = 2.2$, $\alpha = 0$ and $\beta = 0$, while $\mu = 1.28$, 1.58 and 1.78, with correspondingly the eigenvalue $s = 2.9630$, 3.1061 and 3.2171, respectively.

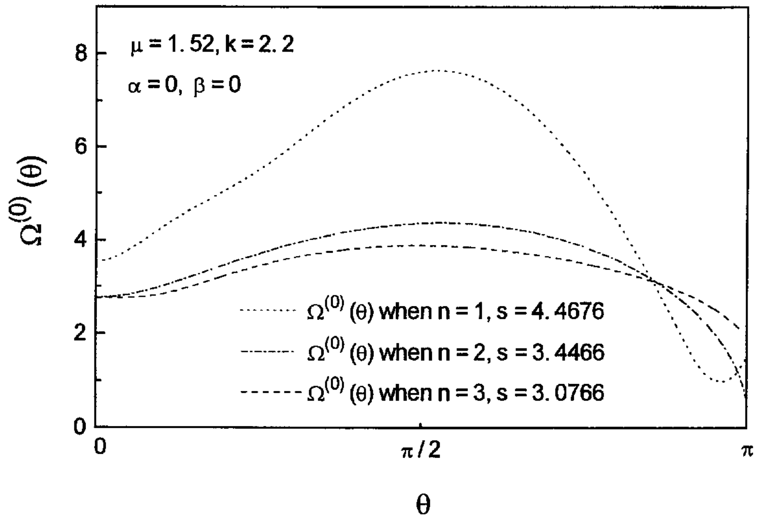


Fig. 7. The angular distributions of $\Omega^{(0)}(\theta)$ when $\mu = 1.52$, $k = 2.2$, $\alpha = 0$ and $\beta = 0$, while $n = 1$, 2 and 3, with correspondingly the eigenvalue $s = 4.4676$, 3.4466 and 3.0766, respectively.

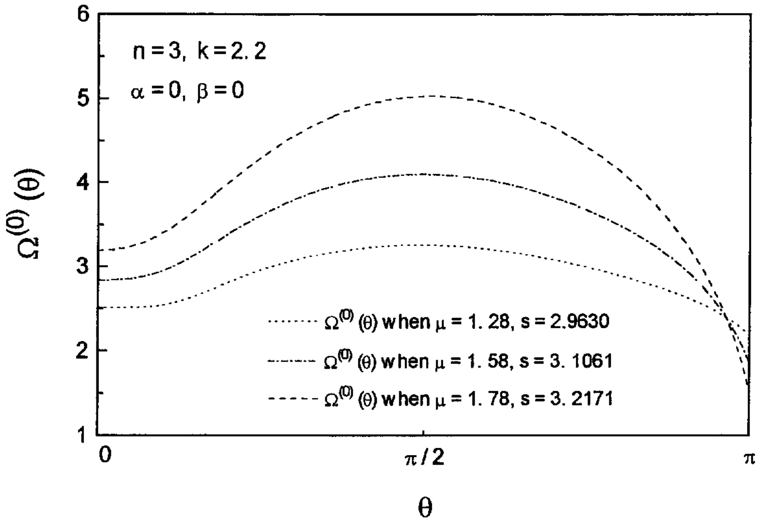


Fig. 8. The angular distributions of $\Omega^{(0)}(\theta)$ when $n = 3$, $k = 2.2$, $\alpha = 0$ and $\beta = 0$, while $\mu = 1.28$, 1.58 and 1.78, with correspondingly the eigenvalue $s = 2.9630$, 3.1061 and 3.2171, respectively.

that of HRR field, where stresses approach a singularity field with the order of $1/(n+1)$ at the crack tip.

It should be pointed out that the parametric distribution of eigenvalue $s$ with the material constants $n$, $k$ and $\mu$ is not uniformly continuous, namely, for some value of these material constants the eigensolution may not exist and discussion regarding this problem has been fulfilled in another work that will be disclosed soon. Besides, like that in the computation of the HRR problem, for some specific values of $n$, $k$ and $\mu$ the third order derivative of eqn (52) at a certain locus changes greatly and therefore the computations become stiff. In this situation one should use either smaller stepsize, if adopting a conventional Runge–Kutta scheme (for instance, the stepsize of $\pi/12{,}000$ is recommended in this paper), or some techniques for dealing with stiff problems.

## 5. CONCLUDING REMARKS

There have been several ways of explaining and erasing the anomaly of stress singularity at a crack tip: introducing the yielding surface of plasticity and, for instance, considering the local large strains at the tip or using the concept of damage, etc. In studying the interaction between crack and damage in the field near the crack tip, one may of course start from continuum damage mechanics and use pure numerical approaches (say, finite element methods) to analyze it. However, numerical procedures do not work so easily because of the localization of damage at its critical points. The problem is no longer elliptic and a local bifurcation of the solution may occur. Thus, in terms of a combined approach of fracture and damage mechanics we employ an analytical procedure to obtain the locally asymptotic structure of the solution that applies to the regime very near the crack tip. Within this regime the damage effect dominates the local zone. According to the solution, stresses vanish with $r \to 0$ due to the damage relaxation to them. Consequently, no stress singularity occurs and stresses approach zero according to $r^{(s-2)}$, where $s$ is a positive eigenvalue larger than 2.

It should be pointed out that in this paper we use a total deformation theory and the infinitesimal strain description. This starting point can be further improved since in the very close region to the crack tip, where damage is extremely severe, the small deformation description is not exactly sound. Nevertheless, the results at which we have arrived provide a new, approximately and asymptotically, of course, analytic representation for the crack tip behavior and allow one to get a step toward the understanding of this problem in a different way.

To accurately determine the amplitudes of stress or strain, which are time and loading dependent, one must, as stated previously, know the global solution of the problem under study. As a global analytical solution is actually impossible to attain, one has to use some numerical or approximate methods to solve this problem. For instance, the approaches proposed in some previous studies in dealing with HRR or RR field (e.g. Bassani and McClintock, 1981; Ainsworth and Budden, 1990; Bassani *et al.*, 1989 and Busso *et al.*, 1995) can be employed to give some hints and this work is being carried out by us through conducting a pure numeric analysis. Particularly, since there is no singularity at the crack tip the $J$-integral cannot be invoked in this case.

From the result obtained in Part I (Lee *et al.*, 1996) and that in this context, we can have such a pattern: in the small damage region around a crack tip, the HRR type singularity still dominates the solution with damage effect as only a perturbation and in the large damage region where the crack tip is quite approached, the damage controls the tip behavior and thereby no stress singularity occurs. Thus, stresses are expected to distribute in this way: at the crack tip stresses vanish and then increase with the increasing of $r$ and finally they reach their corresponding maximum at some position from the crack tip. After that, stresses decrease with the increasing of $r$ approximately according to the law predicted by HRR solution or the one incorporated with the damage influence. Thus, the "singularity" of stress, in correspondence to the common engineering notation, may be understood as finite in this sense. The damage distribution, on the other hand, reaches its critical state of damage at the crack tip. And apart from the crack tip, a mesovolume element of the medium persists in a partly damaged state. As the loading, or the time, continues to increase and the damage constantly accumulates until this element fully fails. Then the crack commences to grow and the problem becomes a growing crack one to which our analysis does not apply.

*Acknowledgements*—This work is partly supported by the Korea Science and Engineering Foundation and partly by China National Science Foundation.

## REFERENCES

Ainsworth, R. A. and Budden, P. J. (1960) Crack tip field under non-steady creep conditions—1. Estimates of the amplitude of the fields. *Fatigue Fracture Engineering Material Structure* **13,** 263–276.

Bassani, J. L. and McClintock, F. A. (1981) Creep relaxation of stress around a crack tip. *International Journal of Solids and Structures* **17,** 479–492.

Bassani, J. L., Hawk D. E. and Saxena, A. (1989) Evaluation of the $C_t$ parameter for characterizing creep crack growth rate in the transient regime, nonlinear fracture mechanics (eds A. Saxena *et al.*), *ASTM STP* 995, p. 7–26.

Bassani, J. L. and Hawk, D. E. (1990) Influence of damage on crack-tip field under small-scale-creep conditions. *International Journal of Fracture* **42,** 157–172.

Busso, E. P., Dean D. W. and Linkens, D. (1995) On the effects of loading conditions and geometry on time-dependent singular crack tip fields. *Engineering Fracture Mechanics* **50,** 231–247.

Chaboche, J. L. (1988a) Continuum damage mechanics: part I—general concepts. *Journal of Applied Mechanics* **55,** 59–64.

Chaboche, J. L. (1988b) Continuum damage mechanics: part II—damage growth, crack initiation and crack growth. *Journal of Applied Mechanics* **55,** 65–72.

Gao, Y. C. (1986) The asymptotic solution to the dynamic crack-tip field in a strain damage material. *International Journal of Engineering Science* **24,** 1045–1055.

Gao, Y. C. (1995) Damage field near a stationary crack tip. *International Journal of Solids and Structures* **32,** 1979–1987.

Hayhurst, D. R. (1972) Creep rupture under multi-axial states of stress. *Journal of Mechanics and Physics of Solids* **20,** 381–390.

Hutchinson, J. W. (1968) Singularity behavior at the end of a tensile crack tip in a hardening material. *Journal of Mechanics and Physics of Solids* **16,** 13–31.

Kachanov, L. M. (1986) *Introduction to Continuum Mechanics.* Martinus Nijhoff, The Netherlands.

Lee, S. B., Lu, M. and Kim, J. Y. (1996) An asymptotic analysis to a tensile crack in creeping solids coupled with damage—part I. small damage region around the crack tip. *International Journal of Solids and Structures* (accepted).

Li, F. Z., Needleman, A. and Shih, C. F. (1988) Creep crack growth by grain boundary cavitation: crack tip field and crack growth rates under transient conditions. *International Journal of Fracture* **38,** 241–273.

McMeeking, R. M. and Parks, D. M. (1979) On critical condition for *J*-dominance of crack tip fields in large-scale yielding. *ASTM STP* 668. American Society for Testing and Materials, Philadelphia, PA, pp. 175–194.

Rabotnov, Y. N. (1969) *Creep Problems in Structural Members.* North-Holland, Amsterdam.

Rice, J. R. and Rosengren, G. F. (1968) Plane strain deformation near a crack tip in a power law hardening material. *Journal of Mechanics and Physics of Solids* **16,** 1–12.

Riedel, H. and Rice, J. R. (1980) Tensile cracks in creeping solids. *Fracture Mechanics*: 12*th Conference, ASTM STP* 700. American Society of Testing and Materials, Philadelphia, PA, pp. 112–130.